positions for which the interceptor flies straight ahead to intercept the target, for which the interceptor starts out with its bank angle at half the maximal value, and with the starting bank just saturated, respectively. In Figs. 1a and 1b, the isochrone sections for τ_f varying 5-20 s terminate on a Barrier section, because the reachability condition[3] becomes an equality. However, the interceptor has a maximum speed capability that is higher than the target speed, and so it can accelerate and eventually capture the target. This is reflected by the discontinuity in the time-to-capture across the Barrier section.

Conclusion

Sections of the feedback solution for interception in a horizontal plane have been mapped by constructing isochrones from open-loop extremals of the interceptor and the target. Both a passive and an optimally evading target were considered. Construction of the feedback solution requires little additional computational effort over that needed to compute the open-loop extremals, and the feedback solution should be capable of onboard implementation.

The aircraft dynamic models employed in the analysis are realistic in terms of aerodynamic forces and constraints, but the restriction to a horizontal plane makes the feedback solutions of limited practical value. The horizontal plane analysis, however, is a necessary step toward developing feedback solutions for the three-dimensional problem.

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An Elementary Proof of the Optimality of Hohmann Transfers

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Introduction

A GRADIENT method is used to prove analytically that the Hohmann transfer is the optimal two-impulse transfer between coplanar circular orbits in a central force field with a Newtonian attraction. The proof is elementary in that only well-known properties of conics are assumed. By an extension of the method, it is shown that a Hohmann type transfer is the optimal two-impulse transfer between coplanar orbits, one of which is circular and the other elliptical.

The Hohmann transfer was proposed in Ref. 1. Barrar[2] gives analytic proofs of several results on optimal transfer between coplanar orbits. Marec[3] gives a proof by graphical construction of the optimality of the Hohmann transfer between coplanar circular orbits. Lawden[4] uses extensively the calculus of variations to find solutions to these transfer problems.

The gradient method is distinct from those above. It differs from a critical point method in that it shows clearly that the characteristic velocity attains a minimum on the boundary of the region where the transfer orbits lie.

The gradient method can be applied to show that the bielliptic transfer is the optimal transfer among all three-impulse transfers between coplanar circular orbits. Furthermore, the method can also be used to show that the multielliptic transfer is the optimal transfer among all N-impulse transfers between coplanar circular orbits for $N>3$.

Optimality of the Hohmann Transfer Between Coplanar Circular Orbits

The transfer orbit is a conic that is represented by $p/r=1+e\cos\theta$ where $e\geq 0$ is the eccentricity, $2p$ is the length of the latus rectum and r is the distance from the focus. Units may be chosen so that the gravitational constant is equal to 1. An orbit with parameters (p,e) has associated with it an energy $(e^2-1)/2p$ and an angular momentum $p^{1/2}$. A change in velocity from a circular orbit of radius R to a conic with parameters (p,e) has a magnitude ΔV given by

$$(\Delta V)^2=v^2+v_c^2-2v_c v_\theta$$

where

$$v^2=2/R+(e^2-1)/p,\quad v_c^2=1/R \quad \text{and} \quad v_\theta=p^{1/2}/R$$

By introducing new variables $x=p^{-1/2}$ and $y=ep^{-1/2}$ the magnitude ΔV can be written as

$$(\Delta V)^2=3/R+y^2-x^2-2/R^{3/2}x$$

Let the radii of the circular orbits be R_1 and R_2 with $R_1<R_2$. Let $\mathcal{R}$ denote the region

$$\{x>0,\ y>0 \text{ and } x^2-xy\leq R_2^{-1}<R_1^{-1}\leq x^2+xy\}$$

The requirement that $x^2+xy\geq R_1^{-1}$ be satisfied means that the periapsis of the transfer conic lies within distance R_1 of the focus; the requirement that $x^2-xy\leq R_2^{-1}$ be satisfied means that the apoapsis lies at a distance from the focus not less than R_2. The region $\mathcal{R}$ has a corner (x_0,y_0) at the intersection of the curves defined by $x^2-xy=R_2^{-1}$ and $x^2+xy=R_1^{-1}$. At the corner the values of x_0 and y_0 are

$$x_0=[(R_1+R_2)/2R_1R_2]^{1/2}$$

$$\text{and}\quad y_0=(R_2-R_1)/[2R_1R_2(R_1+R_2)]^{1/2}$$

Figure 1 shows the region $\mathcal{R}$. The diagonal divides $\mathcal{R}$ into two subregions of ellipses (I), $y<x$, and hyperbolas (II), $y>x$. The diagonal represents parabolas for which $e=1$.

We denote the characteristic velocity $\Delta V_1+\Delta V_2$ of an orbital transfer by V_{CH}. The partial derivative $\partial V_{CH}/\partial y$ is given by $y[(\Delta V_1)^{-1}+(\Delta V_2)^{-1}]$ and is positive throughout $\mathcal{R}$. The gradient $(\partial V_{CH}/\partial x,\ \partial V_{CH}/\partial y)$ of the characteristic velocity is normal to the level curves $V_{CH}=\text{constant}$. The negative gradient of V_{CH} points in the direction of the maximum decrease of V_{CH}. At any interior point of $\mathcal{R}$ the

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derivative $\partial V_{CH}/\partial y > 0$ requires that y must be reduced so that V_{CH} is reduced. Thus, by starting at an interior point of $\mathcal{R}$ and by following the negative gradient one arrives at the boundary of $\mathcal{R}$.

For the remainder of the proof, the gradient method is applied to the restriction of V_{CH} to the boundary of $\mathcal{R}$. Let $\tilde{V}_{CH}$ denote this restricted function. The boundary of $\mathcal{R}$ is the union of two curves C_1 and C_2 defined by C_1: $x \geq x_0$ and $x^2 - xy = R_2^{-1}$, and C_2: $x \leq x_0$ and $x^2 + xy = R_1^{-1}$. By further restricting $\tilde{V}_{CH}$ to C_1 and $\tilde{V}_{CH}$ to C_2 one obtains the characteristic velocity as two distinct functions of x.

On C_1: $y(x) = x - (R_2 x)^{-1}$ holds and on C_2: $y(x) = (R_1 x)^{-1} - x$ holds. Let $\widetilde{\Delta V}_1$ and $\widetilde{\Delta V}_2$ denote restricted functions. Therefore, on C_1 one obtains

$$(\widetilde{\Delta V}_1)^2 = 3/R_1 - 2/R_2 - 2/R_1^{3/2}x + 1/R_2^2 x^2$$

and

$$(\widetilde{\Delta V}_2)^2 = (1/R_2)(1 - 1/R_2^{1/2}x)^2$$

Consequently, the derivative of $\tilde{V}_{CH}$ on C_1 is given by

$$d\tilde{V}_{CH}/dx = x^{-2}[(1/\widetilde{\Delta V}_1)(1/R_1^{3/2} - 1/R_2^2 x) + 1/R_2] > 0$$

for $x \geq x_0$.

On C_2 one obtains

$$(\widetilde{\Delta V}_1)^2 = (1/R_1)(1/R_1^{1/2}x - 1)^2$$

and

$$(\widetilde{\Delta V}_2)^2 = 3/R_2 - 2/R_1 + 1/R_1^2 x^2 - 1/R_2^{3/2}x$$

Consequently, the derivative of $\tilde{V}_{CH}$ on C_2 is given by

$$d\tilde{V}_{CH}/dx = x^{-2}[-1/R_1 + (1/\widetilde{\Delta V}_2)(-1/R_1^2 x + 1/R_2^{3/2})] < 0$$

for $x \leq x_0$.

The negative gradient of $\tilde{V}_{CH}$ on the boundary of $\mathcal{R}$ points toward the corner (x_0, y_0). Thus, V_{CH} attains a global minimum in $\mathcal{R}$ at (x_0, y_0) which represents the Hohmann transfer.

Optimality of the Hohmann Transfer Between a Circular Orbit and an Elliptical Orbit

An extension of the gradient method to orbital transfer between coplanar orbits, one of which is circular and the other elliptical, shows that the Hohmann transfer is optimal.

As in the preceding section, the transfer orbit is a conic that is represented by (x,y). The initial orbit is circular of radius R_1 so that $x_1 = R_1^{-1/2}$ and $y_1 = 0$. The final orbit is elliptical and is represented by (x_2, y_2) where x_2 and y_2 satisfy

$$x_2^2 - x_2 y_2 = R_a^{-1} < R_p^{-1} = x_2^2 + x_2 y_2$$

where R_p and R_a are the distances of periapsis and apoapsis, respectively. The final orbit is chosen to lie outside of the initial orbit so that $R_1 < R_p < R_a$ holds. Finally, let R_2, $R_p \leq R_2 \leq R_a$ denote the distance at which the second impulse is given.

As in the preceding section, the partial derivative by y of the characteristic velocity V_{CH} is always positive in the region $\mathcal{R}$ defined by

$$\{x > 0, y > 0 \quad \text{and} \quad x^2 - xy \leq R_p^{-1} < R_1^{-1} \leq x^2 + xy\}$$

There is a corner of $\mathcal{R}$ at (x_0, y_0). At any interior point of $\mathcal{R}$, $\partial V_{CH}/\partial y > 0$ requires that y must be reduced so that V_{CH} be reduced. One concludes that a global minimum of V_{CH} in $\mathcal{R}$ is attained only on the boundary of $\mathcal{R}$.

By computing the derivative of $\tilde{V}_{CH}$, the restriction of V_{CH} to the boundary of $\mathcal{R}$, one obtains the following two expressions.

On C_1: $x \geq x_0$ and $x^2 - xy = R_2^{-1}$, $d\tilde{V}_{CH}/dx$ is given by

$$d\tilde{V}_{CH}/dx = x^{-2}\{(1/\widetilde{\Delta V}_1)(1/R_1^{3/2} - 1/R_2^2 x) + (1/\widetilde{\Delta V}_1 1/R_2^2)[(2R_pR_a/R_p + R_a)^{1/2} - 1/x]\}$$

For any R_2, $R_p \leq R_2 \leq R_a$, $d\tilde{V}_{CH}/dx > 0$ holds for $x \geq x_0$.

On C_2: $x \leq x_0$ and $x^2 + xy = R_1^{-1}$, $d\tilde{V}_{CH}/dx$ is given by

$$d\tilde{V}_{CH}/dx = x^{-2}\{1/(\widetilde{\Delta V}[1/R_1^{3/2} - 1/R_1^2 x] + 1/\widetilde{\Delta V}_2[1/R_2^2(2R_pR_a)/R_p + R_a)^{1/2} - 1/R_1^2 x]\}$$

For any R_2, $R_p \leq R_2 \leq R_a$, $d\tilde{V}_{CH}/dx < 0$ holds for $x \leq x_0$.

Thus, the minimum of V_{CH} for fixed R_2 is attained at the corner of $\mathcal{R}$ which represents the Hohmann transfer. The absolute minimum of V_{CH} in the class of Hohmann transfers $R_p \leq R_2 \leq R_a$ is attained for $R_2 = R_a$. This can be shown by calculating that $\partial V_{CH}/\partial R_2(x_0, y_0) < 0$ holds.

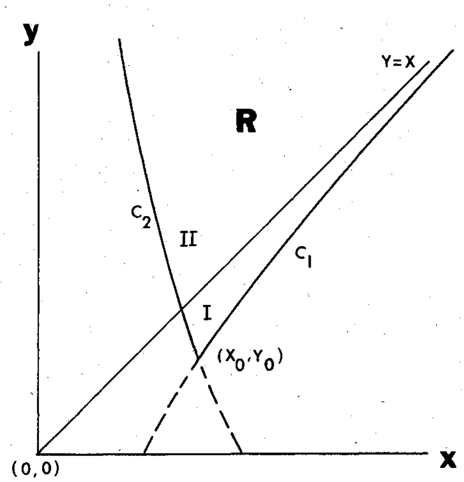

Fig. 1 Region $\mathcal{R}$ for two-impulse transfers.

Conclusion

By carefully choosing coordinates in order to write down the velocity change required to transfer between a circular orbit and a conic, we have shown that the velocity change decreases as one coordinate decreases. Thus, the problem of minimizing the characteristic velocity over the region $\mathcal{R}$ of all admissible transfer conics is transformed immediately into an optimization problem on the boundary of $\mathcal{R}$. When three or more impulses are required for orbital transfer, this procedure reduces by a factor of two the dimension of the optimization problem.

Acknowledgment

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